AUTOMAP:
Inferring Rank-Polymorphic Function Applications with Integer Linear Programming

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- `sqrt [[1,4,9], [16,25,36]] => [[1,2,3], [4,5,6]]`

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- Practically all rank polymorphic languages are **dynamic**: NumPy, APL, MATLAB, ...

`map` and `rep`

- `map f xs` applies `f` to each element of `xs`:

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map f [x_0, x_1, ..., x_n] = [f x_0, f x_1, ..., f x_n]
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   = [x_0 + y_0, ..., x_n + y_n]
  ```

- `rep x` makes an array of unspecified length whose elements are all `x`:

  ```
  rep x = [x, x, ..., x]
  ```

  - We'll ignore the question of how many elements are needed.

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[[1,2],[3,4]] + 1
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yss : [][]int
f: []int -> [][]int -> int
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reps can often be eliminated

```
map (\xs -> f xs yss) xss
```

Goal

For each function application, the compiler should automatically insert `map`s or `rep`s to make the application **rank-correct**.

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$$\texttt{f x} \Longrightarrow \texttt{map (... (map f) ...) (rep ... (rep x) ...)}$$

Challenge: ambiguity

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sum : []int -> int
length : []a -> int
xss : [][]int
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sum (length xss)
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- Never necessary to `map` and `rep` in the same application to obtain a rank-correct program.

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Challenge: elaboration is global

- `sum (length xss)` can be elaborated to:
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     - **Global** reasoning: `length xss` is rank-correct as-is, but a **`map`** is inserted because of the outer `sum` application.
- Elaborations of inner applications affect outer applications.
  - To find all minimal elaborations, must consider all applications **simultaneously**.

Challenge: type variables

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- How do we statically insert **`map`**s and **`rep`**s in the presence of type variables, whose ranks aren't known?

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- We only care about rank, so relax to

$$|p| = |a|$$

where $|p|$ is the **rank** of p.
For example, $|[\,][\,]\texttt{int}| = 2$ and $|\texttt{int}| = 0$.

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    `[1,2,3] : []int`

    Application `sqrt [1,2,3]` gives the constraint

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  - M is equal to the number of `map`s required:

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  - R is equal to the number of `rep`s required:
    - `length (rep 3)`
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    - ...

Constraints - Summary

- Each application of a function `f : p -> c` to an argument `x : a` generates a constraint

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- Rule 1: can either **`map`** or **`rep`** but not both

$$M = 0 \text{ or } R = 0$$

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$$
\begin{aligned}
&\text{minimize} \\
&\qquad M_1 + R_1 + M_2 + R_2 \\
&\text{subject to} \\
&\qquad \left.\begin{array}{l} M_1 + 1 + |a| = R_1 + 2 \\ M_1 = 0 \text{ or } R_1 = 0 \end{array}\right\} \texttt{length} \\
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- Rule 2: Minimize the number of **`map`**s and **`rep`**s
- The or-constraints can be linearized to obtain an integer linear program (ILP).

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1. For each application generate rank equality and Rule 1 (`map` or `rep` but not both) constraints.

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3. Use the ILP solution to elaborate. E.g., if the *i*-th application `f x` has $M_i = 3$ and $R_i = 0$:

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4. Type check elaborated program and continue with compilation as usual.

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- Difficult to quantify value of feature that is glorified syntax sugar.
- We (manually!) rewrote programs to take advantage of AUTOMAP when we judged it improved readability.

Practical impact: before

```
def main [nK][nX]
        (kx: [nK]f32) (ky: [nK]f32) (kz: [nK]f32)
        (x: [nX]f32) (y: [nX]f32) (z: [nX]f32)
        (phiR: [nK]f32) (phiI: [nK]f32)
      : ([nX]f32, [nX]f32) =
  let phiM = map (\r i -> r*r + i*i) phiR phiI
  let as = map (\x_e y_e z_e ->
          map (2*pi*)
            (map (\kx_e ky_e kz_e ->
              kx_e*x_e + ky_e*y_e + kz_e*z_e)
             kx ky kz))
         x y z
  let qr = map (\a -> sum(map2 (*) phiM (map cos a))) as
  let qi = map (\a -> sum(map2 (*) phiM (map sin a))) as
  in (qr, qi)
```

Practical impact: after

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def main [nK][nX]
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         (phiR: [nK]f32) (phiI: [nK]f32)
       : ([nX]f32, [nX]f32) =
  let phiM = phiR*phiR + phiI*phiI
  let as = 2*pi*(kx*transpose (rep x)
             + ky*transpose (rep y)
             + kz*transpose (rep z))
  let qr = sum (cos as * phiM)
  let qi = sum (sin as * phiM)
  in (qr, qi)
```
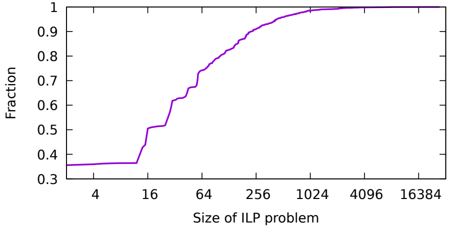

Metrics from changing a benchmark suite

Proportion of ILP problems that have less than some given number of constraints.



Number of programs: 67

Lines of code: 8621 ⇒ 8515

Change in `maps`: 467 ⇒ 213

Largest ILP size: 28104 constraints

Median ILP size: 16 constraints

Mean ILP size: 116 constraints

Mean type checking slowdown: 2.50×

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- Anything inferred can also be inserted explicitly (much like classic type systems!)
- Type checking based on some heavy machinery (ILP), but we suspect of a fairly simple kind.
- Implemented in Futhark, but not really production ready yet.
  - TODO: quality of type errors, type checking speed, better ambiguity checking.

That's it!

- Check out Futhark: https://futhark-lang.org
  - There's a blog post on AUTOMAP that covers this talk in more detail.
  - The paper with a full formalization can also be found there.
- These slides and more about me at https://rschenck.com.