AUTOMAP:
Inferring Rank-Polymorphic Function Applications with Integer Linear Programming

Robert Schenck [1], Nikolaj Hey Hinnerskov [1], Troels Henriksen [1], Magnus Madsen [2], Martin Elsman [1]

[1]DIKU
University of Copenhagen
Denmark

[2]Aarhus University
Denmark

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Context

This presentation is about **Futhark**, a **functional** array language.

- Designed to study compilation of array languages.
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- Designed to study compilation of array languages.
- Space is function application: `f x` means $f(x)$.
- Functions are **curried**: `f x y z` means `((f x) y) z`.
- Hindley-Milner style **static type system** (it has parametric polymorphism).

Example

```
def dotprod x y = sum (map (*) x y)
```

Rank polymorphism

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- `[1,2,3] + 4 => [5,6,7]`
- `sqrt [[1,4,9], [16,25,36]] => [[1,2,3], [4,5,6]]`

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- Practically all rank polymorphic languages are **dynamic**: NumPy, APL, MATLAB, ...

`map` and `rep`

- `map f xs` applies `f` to each element of `xs`:

```
map f [x_0, x_1, ..., x_n] = [f x_0, f x_1, ..., f x_n]
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  = [x_0 + y_0, ..., x_n + y_n]
```

- `rep x` makes an array of unspecified length whose elements are all `x`:

```
rep x = [x, x, ..., x]
```

  - We'll ignore the question of how many elements are needed.

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[[1,2],[3,4]] + 1
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xss : [][]int
f: []int -> [][]int -> int
f xss xss
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rep can often be free by fusing it into function

```
map (\xs -> f xs xss) xss
```

Goal

For each function application, the compiler should automatically insert **`map`**s and **`rep`**s to make the application **rank-correct**.

```
f x  ⟹  map ( ... (map f) ... ) (rep ... (rep x))
```

Challenge: ambiguity

Consider

```
sum: []int -> int
length : []a -> int
xss : [][]int

sum (length xss)
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Many rank-correct elaborations:

1. `sum (`**`rep`**` (length xss))`

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1. `sum (rep (length xss))`
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Many rank-correct elaborations:

1. `sum (rep (length xss))`
2. `sum (map length xss)`
3. `map sum (map (map length) (rep xss))`
4. ...

The Strategy

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An application can be `map`ped or `rep`ped (or neither) but **never both**.

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  BAD:
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  - `(map (map g)) (rep x)`

- Never necessary to **`map`** and **`rep`** in the same application to obtain a rank-correct program.

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- Generally aligns with programmer's intent/simple mental model.
- Minimization over **all** the applications of a top-level definition:
  - Only have to choose from the set of minimal solutions.
    `sum (length xss)` can be elaborated to:
    1. `sum (rep (length xss))`
    2. `sum (map length xss)`

Challenge: elaboration is global

- `sum (map length xss)` is a **global minimal** elaboration of `sum (length xss)`.
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- To find minimal elaborations, must consider all applications **simultaneously**.

Challenge: type variables

- Futhark has parametric polymorphism:

```
id : a -> a
length : []a -> int
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- A type variable can have any rank!

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- A type variable can have any rank!

- How do we statically insert **`map`**s and **`rep`**s in the presence of type variables, whose ranks aren't known?

Constraints

- Suppose

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f : p -> b
x : a
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- The application `f x` has constraint

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- For example: $|$**[ ][ ]**`int`$| = 2$ and $|$`int`$| = 0$.

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  - Example:

```
sqrt : int -> int
[1,2,3] : []int
```

Application `sqrt [1,2,3]` gives the constraint

$$M + \underbrace{|\texttt{int}|}_{0} = \underbrace{|\texttt{[]int}|}_{1} \implies M = 1$$

M is equal to the number of **`map`**s required: **`map`** `sqrt [1,2,3]`

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  - Example: `length : []a -> int` The application `length 3` gives the constraint

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$$|[\,]\,a| = R + |\texttt{int}|$$
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  R is equal to the number of **`rep`**s required:

    - `length (rep 3)`
    - `length (rep (rep 3))`
    - ...

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- Rule 1: can either **`map`** or **`rep`** but **not both**

$$M = 0 \text{ or } R = 0$$

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- Example: `sum (length xss)`

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$$M_1 + R_1 + M_2 + R_2$$

subject to

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- Rule 2: Minimize the number of **`map`**s and **`rep`**s
- The or-constraints can be linearized to obtain an **Integer Linear Program (ILP)**.

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3. Use ILP solution to elaborate. E.g., if the *i*-th application `f x` has $M_i = 3$ and $R_i = 0$:

$$\texttt{f x} \quad \Longrightarrow \quad \texttt{\textbf{map} (\textbf{map} (\textbf{map} f)) x}$$

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4. Type check elaborated program and continue with compilation.

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Practical impact

- Difficult to quantify value of feature that is glorified syntax sugar.
- We (manually!) rewrote programs to take advantage of AUTOMAP when we judged it improved readability.

Practical impact: before

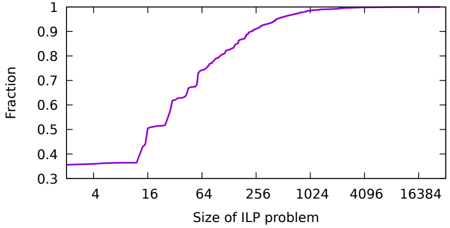
```
def main [nK][nX]
         (kx: [nK]f32) (ky: [nK]f32) (kz: [nK]f32)
         (x: [nX]f32) (y: [nX]f32) (z: [nX]f32)
         (phiR: [nK]f32) (phiI: [nK]f32)
       : ([nX]f32, [nX]f32) =
  let phiM = map (\r i -> r*r + i*i) phiR phiI
  let as = map (\x_e y_e z_e ->
                 map (2*pi*)
                   (map (\kx_e ky_e kz_e ->
                      kx_e*x_e + ky_e*y_e + kz_e*z_e)
                     kx ky kz))
              x y z
  let qr = map (\a -> sum(map2 (*) phiM (map cos a))) as
  let qi = map (\a -> sum(map2 (*) phiM (map sin a))) as
  in (qr, qi)
```

Practical impact: after

```
def main [nK][nX]
         (kx: [nK]f32) (ky: [nK]f32) (kz: [nK]f32)
         (x: [nX]f32) (y: [nX]f32) (z: [nX]f32)
         (phiR: [nK]f32) (phiI: [nK]f32)
       : ([nX]f32, [nX]f32) =
  let phiM = phiR*phiR + phiI*phiI
  let as = 2*pi*(kx*transpose (rep x)
             + ky*transpose (rep y)
             + kz*transpose (rep z))
  let qr = sum (cos as * phiM)
  let qi = sum (sin as * phiM)
  in (qr, qi)
```

Metrics from changing a benchmark suite

Proportion of ILP problems that have less than some given number of constraints.



Number of programs: 67

Lines of code: 8621 ⇒ 8515

Change in `maps`: 467 ⇒ 213

Largest ILP size: 28104 constraints

Median ILP size: 16 constraints

Mean ILP size: 116 constraints

Mean type checking slowdown: 2.50×

Related work

- **Typed Remora**:
  - Very general/powerful; binds shape variables in types: $\forall S.S\ \texttt{int} \to \texttt{int}$.
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- **Typed Remora**:
  - Very general/powerful; binds shape variables in types: $\forall S.S$ `int` $\rightarrow$ `int`.
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- **Naperian Functors** (Jeremy Gibbons):
  - Cool rank polymorphism encoding in Haskell.
  - Complicated function types (and potentially error messages).
- **Single-assignment C**:
  - Has *rank specialization* where functions have specialized definitions depending on the rank of the input.
  - No parametric polymorphism or higher-order functions.

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Summary

- AUTOMAP is a conservative extension of/compatible with a Hindley-Milner type system for array programming.
- Anything inferred can also be inserted explicitly (much like classic type systems!)
- Type checking based on some heavy machinery (ILP), but we suspect of a fairly simple kind.
- Implemented in Futhark, but not really production ready yet.
  - Todo: quality of type errors, type checking speed.

- Check out Futhark: https://futhark-lang.org
  - There's a blog post on AUTOMAP that covers this talk in more detail. (A paper is coming too.)

- Check out me: https://rschenck.com
  - Graduating December 2024, **looking for a postdoc** (or a real job) in types/functional programming/compilers.

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Can we always rewrite `map f x` as `f x`?

- Consider (outer product)

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- Removing the outer `map`:

```
(\x -> x * ys) xs
```

(which is the same as `xs * ys`). This is elaborated by AUTOMAP to

```
map (*) xs ys
```