AUTOMAP:
Inferring Rank-Polymorphic Function Applications with Integer Linear Programming

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Example

```
def dotprod x y = sum (map (*) x y)
```

Rank polymorphism

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- `sqrt [[1,4,9], [16,25,36]] => [[1,2,3], [4,5,6]]`

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`map (+) [1,2,3] [4,5,6]` vs. `[1,2,3] + [4,5,6]`

- Practically all rank polymorphic languages are **dynamic**: NumPy, APL, MATLAB, ...

`map` and `rep`

- `map f xs` applies `f` to each element of `xs`:

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map f [x_0, x_1, ..., x_n] = [f x_0, f x_1, ..., f x_n]
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  ```

- `rep x` makes an array of unspecified length whose elements are all `x`:

  ```
  rep x = [x, x, ..., x]
  ```

  - We'll ignore the question of how many elements are needed.

An example

```
[[1,2],[3,4]] + 1
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[[1,2],[3,4]] + rep (rep 1)
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f: []int -> [][]int -> int
f xss xss
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because

```
(map f xss) : [][][]int -> []int
```

reps can often be eliminated

```
map (\xs -> f xs xss) xss
```

Goal

For each function application, the compiler should automatically insert `map`s or `rep`s to make the application **rank-correct**.

$$\texttt{f x} \Longrightarrow \begin{cases} \texttt{map (... (map f) ...) x} \\ \text{or} \\ \texttt{f (rep ... (rep x))} \end{cases}$$

Challenge: ambiguity

Consider

```
sum: []int -> int
length : []a -> int
xss : [][]int

sum (length xss)
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1. `sum (`**`rep`**` (length xss))`

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1. `sum (rep (length xss))`
2. `sum (map length xss)`
3. `map sum (map (map length) (rep xss))`
4. ...

The Strategy

Rule 1

An application can be `map`ped or `rep`ped (or neither) but **never both**.

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- Never necessary to `map` and `rep` in the same application to obtain a rank-correct program.

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- Generally aligns with programmer's intent/simple mental model.
- Minimization over **all** the applications of a top-level definition:
  - Only have to choose from the set of minimal solutions.
    `sum (length xss)` can be elaborated to:
    1. `sum (rep (length xss))`
    2. `sum (map length xss)`

Challenge: elaboration is global

- `sum (map length xss)` is a **global minimal** elaboration of `sum (length xss)`.
  - Inserting the **`map`** for the inner `length` application requires considering the outer `sum`.

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- `sum (map length xss)` is a **global minimal** elaboration of `sum (length xss)`.
  - Inserting the **`map`** for the inner `length` application requires considering the outer `sum`.
- To find all minimal elaborations, must consider all applications **simultaneously**.

Challenge: type variables

- Futhark has parametric polymorphism:

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length : []a -> int
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- A type variable can have any rank!

- How do we statically insert **`map`**s and **`rep`**s in the presence of type variables, whose ranks aren't known?

Constraints

- Suppose

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f : p -> b
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- For example: $|\textbf{[ ][ ]}\texttt{int}| = 2$ and $|\texttt{int}| = 0$.

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  - Example:

```
sqrt : int -> int
[1,2,3] : []int
```

Application `sqrt [1,2,3]` gives the constraint

$$M + \underbrace{|\texttt{int}|}_{0} = \underbrace{|\texttt{[]int}|}_{1} \implies M = 1$$

M is equal to the number of **`map`**s required: **`map`** `sqrt [1,2,3]`

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  - Example: `length : []a -> int` The application `length 3` gives the constraint

$$|\mathbf{[\,]}\,a| = R + |\texttt{int}|$$
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$$|[\,]\,a| = R + |\texttt{int}|$$
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  R is equal to the number of **`rep`**s required:
    - `length (rep 3)`
    - `length (rep (rep 3))`
    - ...

Constraints

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- **Rule 1**: can either **`map`** or **`rep`** but **not both**

$$M = 0 \text{ or } R = 0$$

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- Example: `sum (length xss)`

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- Rule 2: Minimize the number of **`map`**s and **`rep`**s
- The or-constraints can be linearized to obtain an **Integer Linear Program (ILP)**.

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3. Use ILP solution to elaborate. E.g., if the *i*-th application `f x` has $M_i = 3$ and $R_i = 0$:

$$\texttt{f x} \quad \Longrightarrow \quad \texttt{\textbf{map} (\textbf{map} (\textbf{map} f)) x}$$

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4. Type check elaborated program and continue with compilation.

Formalization

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\begin{array}{rcll}
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e & ::= & v \\
 & \mid & x \quad (x \in V_{\mathrm{p}}) \\
 & \mid & [e, \ldots, e] \\
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$$

- The application $e\ e \bigtriangleup (M, R)$ is a *flexible function application*.

Flexible function applications

- During constraint generation, all function applications $f\ x$ are annotated with rank variables:

$$f\ x \quad \longrightarrow \quad f\ x \bigtriangleup (\color{blue}{M}, \color{red}{R})$$

Flexible function applications

- During constraint generation, all function applications $f\ x$ are annotated with rank variables:

$$f\ x \quad \longrightarrow \quad f\ x \bigtriangleup (M, R)$$

- $f\ x \bigtriangleup (M, R)$ will ultimately be elaborated to

$$\mathbf{map}^{s_{\mathrm{r}}(M)}\ f\ (\mathbf{rep}^{s_{\mathrm{r}}(R)}\ x)$$

  where s_{r} is the solution to the ILP and

$$\begin{aligned} \mathbf{map}^0\ e &= e \\ \mathbf{map}^{n+1}\ e &= \mathbf{map}\ (\mathbf{map}^n\ e) \end{aligned} \qquad \begin{aligned} \mathbf{rep}^0\ e &= e \\ \mathbf{rep}^{n+1}\ e &= \mathbf{rep}\ (\mathbf{rep}^n\ e) \end{aligned}$$

Languages

- The formalization is split up into three languages:

	Source lang.		
Implicit **maps**/**reps**	✓		
Explicit **maps**/**reps**	✓		
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Example	`sqrt` $[1, 2, 3]$	`sqrt` $[1, 2, 3] \triangle (M, R)$	`map` `sqrt` $[1, 2, 3]$

$$\textbf{Source} \xrightarrow{\text{type checking}} \textbf{Internal} \xrightarrow{\text{elaboration}} \textbf{Target}$$

Type checking

- Constraint-based type system; judgements of the form

$$\Gamma \vdash e \ :_{S} \ \sigma \ \| \ C$$

- Under environment Γ, e has scheme σ with *frame* S when the constraints in C are satisfied.

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  - Example:

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  - In general, can think of e as having the type $S\ \sigma$.
  - Example:

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  - Frames are needed for proofs, improved ambiguity checking, and implementation optimizations.

Type checking rules - C-APP

- The C-APP rule types flexible function applications.

$$\frac{\Gamma \vdash e_1 \;:_{S_1}\; \tau_1 \to \tau_2 \;\|\; C_1 \qquad \Gamma \vdash e_2 \;:_{S_2}\; \tau_3 \;\|\; C_2}{}\ \text{C-APP}$$

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$$\dfrac{\Gamma \vdash e_1 \;:_{S_1}\; \tau_1 \to \tau_2 \;\|\; C_1 \qquad \Gamma \vdash e_2 \;:_{S_2}\; \tau_3 \;\|\; C_2 \\ M, R \text{ fresh} \qquad C = \{M \mathrel{\dot\vee} R,\ \ [\,]^{M}\ S_1\ \tau_1 \stackrel{?}{=} [\,]^{R}\ S_2\ \tau_3\}}{}\ \textsc{C-App}$$

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- The generated constraints are relaxed to rank constraints, which are used to form the ILP:

$$[\,]^M\ S_1\ \tau_1 \stackrel{?}{=} [\,]^R\ S_2\ \tau_3 \longrightarrow M + |S_1| + |\tau_1| \stackrel{?}{=} R + |S_2| + |\tau_3|$$

Type checking rules - C-Def

- The C-Def rule types top-level polymorphic definitions and dispatches the constraint set for each top-level definition

$$\frac{\Gamma, x : \tau \vdash e \;:\; \tau' \;\|\; C \qquad s \text{ satisfies } C \qquad \{\vec{\alpha}\} \cap \mathbf{ftv}(s(\Gamma), \sigma) = \emptyset \qquad s(\Gamma), f : \forall \vec{\alpha}. s(\tau) \to s(\tau') \vdash p \;:\; \sigma}{s(\Gamma) \vdash \texttt{def}\; f\; x = s(e)\; ;\; p \;:\; \sigma}\;\text{C-Def}$$

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  - s also substitutes type variables.

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input : A constraint set C.
output: A satisfying substitution s.

$|C| \leftarrow$ construct the associated rank constraint set from C;
$I \leftarrow$ construct the corresponding ILP from $|C|$;
$s_{\mathrm{r}} \leftarrow$ solve I using an ILP solver;
if s_{r} **then**
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    if s'_{r} **then**
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Properties of SOLVE

- All satisfiers s can be decomposed into a type and rank substitution:

Proposition

If s satisfies C, there exists a rank substitution s_{r} that satisfies $|C|$ and there exists a closed type substitution s_{t} such that $s|_{\mathrm{ftv}(C)\cup\mathrm{frv}(C)} = s_{\mathrm{t}} \circ s_{\mathrm{r}}$.

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- You can always build a satisfier $s_{\mathrm{t}} \circ s_{\mathrm{r}}$ of C from a satisfier s_{r} of $|C|$:

Proposition

If C is satisfiable and s_{r} satisfies $|C|$ then there is a closed type substitution s_{t} such that the substitution $s = s_{\mathrm{t}} \circ s_{\mathrm{r}}$ satisfies C.

Elaboration

- Reminder: a satisfying substitution s for a constraint set C (which SOLVE gives us) instantiates all rank variables with integral ranks:

$$s(\texttt{sqrt}\ [1,2,3] \bigtriangleup (M, R)) \longrightarrow \texttt{sqrt}\ [1,2,3] \bigtriangleup (1, 0)$$

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- The AM transformation converts from the internal language to the target language:

$$\begin{array}{lcl}
\mathrm{AM}([e_1,\ldots,e_m]) & = & [\mathrm{AM}(e_1),\ldots,\mathrm{AM}(e_m)] \\
\mathrm{AM}(\texttt{def}\ f\ x = e\ ;\ p) & = & \texttt{def}\ f\ x = \mathrm{AM}(e)\ ;\ \mathrm{AM}(p) \\
\mathrm{AM}(e_1\ e_2 \bigtriangleup (n_{\mathrm{M}}, n_{\mathrm{R}})) & = & \boxed{\textbf{map}^{n_{\mathrm{M}}}\ \mathrm{AM}(e_1)\ (\textbf{rep}^{n_{\mathrm{R}}}\ \mathrm{AM}(e_2))} \\
 & \vdots &
\end{array}$$

AM Properties

- The AM transformation preserves well-typedness of programs.

Proposition: Well-typedness

If $\Gamma \vdash e \;:_S\; \sigma \;\|\; C$ and s is a satisfier of C, then $s(\Gamma) \vdash \mathrm{AM}(s(e)) \;:\; s(S\,\sigma)$.

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If the programmer inserts an otherwise inferred `map` or `rep` operation then the resulting program is unambiguous and its elaboration is semantically equivalent to the elaboration of the original program.

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$\uparrow$	$\uparrow$	$\uparrow$
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- Aside from size inference, works very similarly to the SOLVE algorithm, with a couple of exceptions.

Implementation - induced **reps**

- $\texttt{map}\ (\lambda y.\, xs \ast y)\ ys$ can be elaborated to

$$\texttt{map}\,(\lambda y.\,\textbf{map}\,(\ast)\,xs\,(\textbf{rep}\,y))\,ys \quad (1) \quad \text{or} \quad \texttt{map}\,(\lambda y.\,\textbf{map}\,(\ast)\,xs\,y)\,(\textbf{rep}\,ys) \quad (2)$$

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- The `rep` in (1) is **induced** by the outer **map**. It will be eliminated by `rep` fusion:

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- $\texttt{map}\,(\lambda y.\,\texttt{\textbf{map}}\,(\lambda x.\,x \ast y)\,xs)\,ys$ has size **1**.
- $\Rightarrow$ we can disambiguate by only counting **non-induced** $\texttt{\textbf{rep}}$S.

Implementation - induced `rep`s

- Frames track the number of `map`s in an application ($S = [\,]^M S_1$):

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- ILP objective becomes

$$\dots + M + \max(0, |R| - |S|) + \dots$$

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Practical impact

- Difficult to quantify value of feature that is glorified syntax sugar.
- We (manually!) rewrote programs to take advantage of AUTOMAP when we judged it improved readability.

Practical impact: before

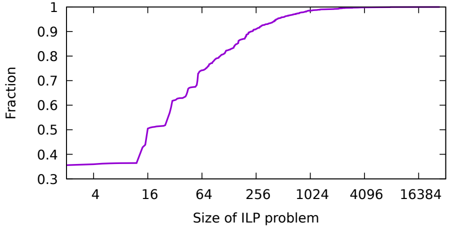
```
def main [nK][nX]
         (kx: [nK]f32) (ky: [nK]f32) (kz: [nK]f32)
         (x: [nX]f32) (y: [nX]f32) (z: [nX]f32)
         (phiR: [nK]f32) (phiI: [nK]f32)
       : ([nX]f32, [nX]f32) =
  let phiM = map (\r i -> r*r + i*i) phiR phiI
  let as = map (\x_e y_e z_e ->
          map (2*pi*)
            (map (\kx_e ky_e kz_e ->
              kx_e*x_e + ky_e*y_e + kz_e*z_e)
             kx ky kz))
         x y z
  let qr = map (\a -> sum(map2 (*) phiM (map cos a))) as
  let qi = map (\a -> sum(map2 (*) phiM (map sin a))) as
  in (qr, qi)
```

Practical impact: after

```
def main [nK][nX]
         (kx: [nK]f32) (ky: [nK]f32) (kz: [nK]f32)
         (x: [nX]f32) (y: [nX]f32) (z: [nX]f32)
         (phiR: [nK]f32) (phiI: [nK]f32)
       : ([nX]f32, [nX]f32) =
  let phiM = phiR*phiR + phiI*phiI
  let as = 2*pi*(kx*transpose (rep x)
          + ky*transpose (rep y)
          + kz*transpose (rep z))
  let qr = sum (cos as * phiM)
  let qi = sum (sin as * phiM)
  in (qr, qi)
```

Metrics from changing a benchmark suite

Proportion of ILP problems that have less than some given number of constraints.



Number of programs: 67

Lines of code: 8621 ⇒ 8515

Change in `maps`: 467 ⇒ 213

Largest ILP size: 28104 constraints

Median ILP size: 16 constraints

Mean ILP size: 116 constraints

Mean type checking slowdown: 2.50×

Related work

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  - Inference is very difficult.

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- **Single-assignment C**:
  - Has *rank specialization* where functions have specialized definitions depending on the rank of the input.
  - No parametric polymorphism or higher-order functions.

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- AUTOMAP is a conservative extension of/compatible with a Hindley-Milner type system for array programming.
- Anything inferred can also be inserted explicitly (much like classic type systems!)
- Type checking based on some heavy machinery (ILP), but we suspect of a fairly simple kind.
- Implemented in Futhark, but not really production ready yet.
  - Todo: quality of type errors, type checking speed, alternate ambiguity checking.

- Check out Futhark: https://futhark-lang.org
  - There's a blog post on AUTOMAP that covers much of this talk in more detail.
  - An AUTOMAP paper will be published at OOPSLA 2024, preprint available at https://rschenck.com.